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On the stability of plane Poiseuille flow with a finite conductivity in an aligned magnetic field

By SUNG-HWAN KO
General Dynamics Corporation Electronics Division,
Rochester, New York†

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A study is made of the stability of a viscous, incompressible fluid with a finite conductivity flowing between parallel planes in a parallel magnetic field. The general form of the magnetohydrodynamic stability equation is a sixth-order differential equation. The complete sixth-order differential equation is solved numerically as an eigenvalue problem. Stability curves are obtained for a range of values of the magnetic Reynolds number R_m and the Alfvén number A based on two-dimensional disturbances. It is found that the minimum critical Reynolds number is raised as R_m increases for a given A^2 and as A^2 increases for a given R_m, respectively. The stability curve closes and finally degenerates to a point which gives the critical value for R_m or A^2. Results obtained for two-dimensional disturbances are modified to take into account three-dimensional disturbances. Then the minimum critical Reynolds number where three-dimensional disturbances become apparent is obtained, below which two-dimensional disturbances are the most unstable.

1. Introduction

The stability of a viscous, electrically conducting fluid flow between parallel planes with a coplanar magnetic field was first investigated by Michael (1953) and Stuart (1954). Stuart (1954) considered the case when the magnetic Reynolds number R_m is small. The general form of the stability equation could, for small R_m, be reduced to a fourth-order differential equation in the velocity perturbation v alone. This equation differs only by the additional term q from the Orr–Sommerfeld equation. This quantity q represents the stabilizing influence. Following Stuart (1954), Velikhov (1959), Tarasov (1960) and Hains (1965) also examined the magnetohydrodynamic stability of plane Poiseuille flow with a parallel magnetic field. Hains (1965) used the stability equation for small R_m given by Stuart (1954). However, in his investigation, the eigenvalue problem was solved using a parameter N defined as $N = q/\alpha$, where α is the wavenumber. Drazin (1960) examined some general aspects of the stabilizing influence for the parallel magnetic field with small R_m. Wooler (1961) also studied the stability of a plane parallel flow for small R_m. In his case the magnetic field lies

† Also, Rochester Institute of Technology, Rochester, New York.

in the plane of the fluid flow, but it is not parallel to the flow. He found that three-dimensional disturbances can be the most unstable ones for a magnetic field not parallel to the flow.

The investigations on the stability of parallel flow with parallel magnetic field were based on the assumption that Squire's (1933) theorem could be extended to this particular case. However, Hunt (1966) pointed out that, even when the magnetic field is parallel to the flow, three-dimensional disturbances are, in general, the most unstable. He showed for all values of R_m 'how results obtained for two-dimensional disturbances can be modified to take into account three-dimensional disturbances and thence draw some general conclusions about the stabilizing influence of a parallel magnetic field on a plane parallel flow'.

In our investigation, for a finite value of R_m, the complete magnetohydrodynamic stability equation (a sixth-order ordinary differential equation) was numerically solved as an eigenvalue problem. A special transformation has been made to build up an accurate difference equation for large values of R and finite values of R_m. The solutions were first made for two-dimensional disturbances, and then the region stable for two-dimensional disturbances but unstable for three-dimensional disturbances has been found using the criterion demonstrated by Hunt (1966).

2. Statement of the problem

We consider the magnetohydrodynamic stability of an incompressible, viscous fluid of a finite conductivity flowing between parallel planes. It is assumed that flow under a pressure gradient is practically parallel to the perfectly conducting planes at a distance $2b_*$ apart and that a uniform magnetic field is coplanar with the flow.

The governing magnetohydrodynamic equations in dimensional form may be written as

$$v^*_{k,k} = 0, \tag{2.1}$$

$$B^*_{k,k} = 0, \tag{2.2}$$

$$\rho_*\left(\frac{\partial v^*_i}{\partial t_*} + v^*_k v^*_{i,k}\right) = -p^*_{,i} + \rho_* \nu_* v^*_{i,kk} + \frac{1}{\mu_*}(B^*_{i,k}B^*_k - B^*_{k,i}B^*_k), \tag{2.3}$$

$$\frac{\partial B^*_i}{\partial t_*} - v^*_{i,k}B^*_k + v^*_k B^*_{i,k} - \frac{B^*_{i,kk}}{\mu_*\sigma_*} = 0, \tag{2.4}$$

where ρ_* is the density of fluid, μ_* the permeability, σ_* the conductivity, ν_* the kinematic viscosity, p_* the hydrodynamic pressure, v^*_i the velocity vector and B^*_i the magnetic field vector.

If the dimensional co-ordinate system is chosen such that the main flow and the magnetic field lies in the direction of x_* and the steady velocity and field are functions of y_* only, then the dimensionless form of the disturbances is given by

$$q'_i = q_i(y)\exp\{i(\alpha x + \beta z) - i\alpha c t\}, \tag{2.5}$$

where $q_i(y)$ is the dimensionless complex amplitude function, $\alpha = \alpha_* b_*$ and $\beta = \beta_* b_*$ the dimensionless wave-numbers in the $x = x_*/b_*$ and $z = z_*/b_*$

directions respectively, $t = t_* U_0^*/b_*$ the dimensionless time co-ordinate, U_0^* a characteristic velocity, $c = c_*/U_0^* = c_r + ic_i$ the dimensionless complex phase velocity; the basic flow is said to be stable, neutrally stable, or unstable according to $c_i < 0$, $c_i = 0$ or $c_i > 0$. If equations (2.1), (2.2), (2.3) and (2.4) are perturbed in the manner of (2.5), then the linearized disturbance equations in dimensionless form, as were derived by Michael (1953) and Stuart (1954), become

$$(w-c)\psi - v = -\frac{i}{\alpha R_m}\left(\frac{d^2\psi}{dy^2} - k^2\psi\right), \tag{2.6}$$

$$(w-c)\left(\frac{d^2v}{dy^2} - k^2v\right) - \frac{d^2w}{dy^2}v - A^2\left(\frac{d^2\psi}{dy^2} - k^2\psi\right) = -\frac{i}{\alpha R}\left(\frac{d^4v}{dy^4} - 2k^2\frac{d^2v}{dy^2} + k^4v\right), \tag{2.7}$$

where ψ and v are the complex amplitude functions of the magnetic field vector and the velocity vector respectively in the direction of y-component, $k = (\alpha^2+\beta^2)^{\frac{1}{2}}$ the dimensionless wave-number, $A = B_0^*/U_0^*(\rho_*\mu_*)^{\frac{1}{2}}$ the Alfvén number, $R = U_0^* b_*/\nu_*$ the hydrodynamic Reynolds number, $R_m = \mu_*\sigma_* b_* U_0^*$ the magnetic Reynolds number and $w = 1-y^2$ the basic velocity profile.

Eliminating v from (2.6) and (2.7) yields a sixth-order linear differential equation in terms of ψ alone, which is the complete magnetohydrodynamic stability equation describing the motion of a disturbance travelling at an angle $\theta = \cos^{-1}(\alpha/k)$ to U_0^* and B_0^*. If the angle θ is equal to zero, this equation can be described by an equivalent two-dimensional one. Then, the equivalent two-dimensional disturbance equation in terms of ψ alone may be written as

$$\begin{aligned}(w-c)&\left\{\left(\frac{d^2}{dy^2}-\alpha^2\right)(w-c)\psi + \frac{i}{\alpha R_m}\left(\frac{d^2}{dy^2}-\alpha^2\right)^2\psi\right\}\\ &-\frac{d^2w}{dy^2}\left\{(w-c)\psi + \frac{i}{\alpha R_m}\left(\frac{d^2}{dy^2}-\alpha^2\right)\psi\right\} - A^2\left(\frac{d^2}{dy^2}-\alpha^2\right)\psi\\ &+\frac{i}{\alpha R}\left\{\left(\frac{d^2}{dy^2}-\alpha^2\right)^2(w-c)\psi + \frac{i}{\alpha R_m}\left(\frac{d^2}{dy^2}-\alpha^2\right)^3\psi\right\} = 0.\end{aligned} \tag{2.8}$$

When the conducting fluid flows between two perfectly conducting walls, the boundary conditions at the walls ($y = \pm 1$) can be written as

$$\psi = 0, \tag{2.9}$$

$$\frac{d^2\psi}{dy^2} = 0, \tag{2.10}$$

$$(w-c)\frac{d\psi}{dy} + \frac{i}{\alpha R_m}\left(\frac{d^3\psi}{dy^3} - \alpha^2\frac{d\psi}{dy}\right) = 0. \tag{2.11}$$

3. Eigenvalue problem

In (2.8) the solution to ψ can be separated into symmetrical and anti-symmetrical modes. However, it is generally believed that antisymmetrical disturbances are more unstable than symmetrical ones; thus our analysis is limited to the solution of an even function $\psi(y)$. Therefore, it is necessary to integrate over half the channel if the boundary conditions

$$d\psi/dy = 0, \tag{3.1}$$

$$d^3\psi/dy^3 = 0, \tag{3.2}$$

$$d^5\psi/dy^5 = 0, \tag{3.3}$$

are satisfied at the channel axis ($y = 0$).

Equation (2.8) is a homogeneous sixth-order ordinary differential equation with six homogeneous boundary conditions (2.9) to (3.3). The solution to the magnetohydrodynamic stability equation may be made by direct numerical integration of the equation, in which the value of c is the eigenvalue to be determined for assigned values of α, R, R_m and A^2. We are also interested in determining the critical Reynolds number R_c, below which the basic flow is completely stable for all wave-numbers. By introducing a special transformation

$$\psi = \left(1+\frac{\delta^2}{4}+\frac{\delta^4}{80}+\frac{\delta^6}{20160}\right)g, \tag{3.4}$$

the MHD stability equation (2.8) can be replaced by an accurate finite-difference equation in terms of g. A homogeneous system of linear algebraic equations arising from the sixth-order differential equation with the relevant boundary conditions may be numerically solved by a direct Gaussian elimination as was done by Thomas (1953) in his investigation of the usual hydrodynamic stability of plane Poiseuille flow. However, in his case, only the fourth-order differential equation (Orr–Sommerfeld equation) was solved as an eigenvalue problem for assigned values of α and R.

In the present investigation, Gaussian elimination was made from the wall toward the centre of the channel. For assigned values of α, R, R_m and A^2, the eigenvalues c were guessed and successively approximated until no virtual changes in the eighth decimal place of c were forthcoming.

4. Stability curves

Squire (1933) demonstrated that, in the ordinary hydrodynamic theory, two-dimensional disturbances are more destabilizing than three-dimensional disturbances for two-dimensional parallel flows. However, Hunt (1966) demonstrated that three-dimensional disturbances can be the most unstable for a plane parallel flow with a parallel magnetic field based on a physical point of view. By virtue of the criterion used in his demonstration we can now show how the stability curves obtained for two-dimensional disturbances can be utilized to take account of three-dimensional disturbances. The computed eigenvalues c in the vicinity of the critical Reynolds numbers are listed in table 1.

The neutral stability curves for various R_m and A^2 based on two-dimensional disturbances are shown in figures 1 and 2. The stability curve with constant amplification rates is shown in figure 3a. The amplification rates for the case $R_m = 1$ in association with $A^2 = 0{\cdot}05$ are mapped for details in figure 3b, and also those in the vicinity of the critical Reynolds number are enlarged as shown in figure 3*c*. The critical Reynolds number for this case reads

$$R_c = (29{\cdot}724)^3 = 26{,}262$$

with $\alpha = 0{\cdot}830$ and the computed eigenvalue is $c = 0{\cdot}17654329 + 0{\cdot}00000025\mathrm{i}$.

Figures 4 and 5 can be drawn from the information obtained from figures 1 and 2. In figure 4 the curve C was obtained by making use of the stability curves

		$R_m = 1,\quad A^2 = 0{\cdot}01$	
$\alpha \backslash R$	20^3 (8,000)	22^3 (10,648)	24^3 (13,824)
0·70	0·193643 − 0·014342i	0·185603 − 0·009469i	0·178363 − 0·005475i
0·75	0·201565 − 0·010035i	0·193120 − 0·005555i	0·185513 − 0·002031i
0·80	0·209128 − 0·006566i	0·200282 − 0·002608i	0·192333 + 0·000377i
0·85	0·216331 − 0·004028i	0·207100 − 0·000659i	0·198826 + 0·001767i
0·90	0·223179 − 0·002437i	0·213568 + 0·000316i	0·204957 + 0·002139i
0·95	0·229645 − 0·001781i	0·219639 + 0·000319i	0·210668 + 0·001516i
1·00	0·235686 − 0·002041i	0·225251 − 0·000608i	0·215875 − 0·000063i
1·05	0·241243 − 0·003180i	0·230321 − 0·002442i	0·220461 − 0·002554i
1·10	0·246232 − 0·005156i	0·234738 − 0·005114i	0·224282 − 0·005879i
		$R_m = 1,\quad A^2 = 0{\cdot}05$	
	28^3 (21,952)	30^3 (27,000)	32^3 (32,768)
0·60	0·153681 − 0·014601i	0·148678 − 0·011381i	0·144018 − 0·008640i
0·65	0·160520 − 0·009651i	0·155157 − 0·006829i	0·150200 − 0·004500i
0·70	0·166919 − 0·005800i	0·161263 − 0·003432i	0·156063 − 0·001543i
0·75	0·172978 − 0·003069i	0·167068 − 0·001177i	0·161651 + 0·000253i
0·80	0·178720 − 0·001438i	0·172570 − 0·000040i	0·166939 + 0·000930i
0·85	0·184121 − 0·000873i	0·177721 + 0·000025i	0·171856 + 0·000519i
0·90	0·189117 − 0·001333i	0·182433 − 0·000942i	0·176289 − 0·000934i
0·95	0·193607 − 0·002767i	0·186579 − 0·002891i	0·180081 − 0·003369i
		$R_m = 1,\quad A^2 = 0{\cdot}07$	
	36^3 (46,656)	38^3 (54,872)	40^3 (64,000)
0·60	0·136526 − 0·007026i	0·132679 − 0·005259i	0·129085 − 0·003770i
0·65	0·141954 − 0·003568i	0·137923 − 0·002190i	0·134173 − 0·001067i
0·70	0·147133 − 0·001334i	0·142944 − 0·000343i	0·139052 + 0·000411i
0·75	0·152059 − 0·000269i	0·147710 + 0·000331i	0·143669 + 0·000717i
0·80	0·156664 − 0·000325i	0·152132 − 0·000118i	0·147911 − 0·000109i
0·85	0·160832 − 0·001454i	0·156069 − 0·001648i	0·151609 − 0·002023i
		$R_m = 0{\cdot}01,\quad A^2 = 0{\cdot}05$	
	22^3 (10,648)	24^3 (13,824)	26^3 (17,576)
0·65	0·177944 − 0·017135i	0·171360 − 0·012153i	0·165286 − 0·007995i
0·70	0·186332 − 0·012303i	0·179265 − 0·007647i	0·172769 − 0·003891i
0·75	0·194242 − 0·008281i	0·186727 − 0·004088i	0·179858 − 0·000823i
0·80	0·201723 − 0·005193i	0·193798 − 0·001546i	0·186597 + 0·001167i
0·85	0·208801 − 0·003094i	0·200497 − 0·000053i	0·192978 + 0·002085i
0·90	0·215479 − 0·001993i	0·206803 + 0·000412i	0·198966 + 0·001937i
0·95	0·221729 − 0·001880i	0·212669 − 0·000143i	0·204482 + 0·000742i
1·00	0·227497 − 0·002735i	0·218012 − 0·001686i	0·209416 − 0·001464i
1.05	0·232703 − 0·004521i	0·222721 − 0·004182i	0·213614 − 0·004635i

TABLE 1. Eigenvalues c for MHD stability curves based on two-dimensional disturbances

(Integration interval $h = 0{\cdot}005$)

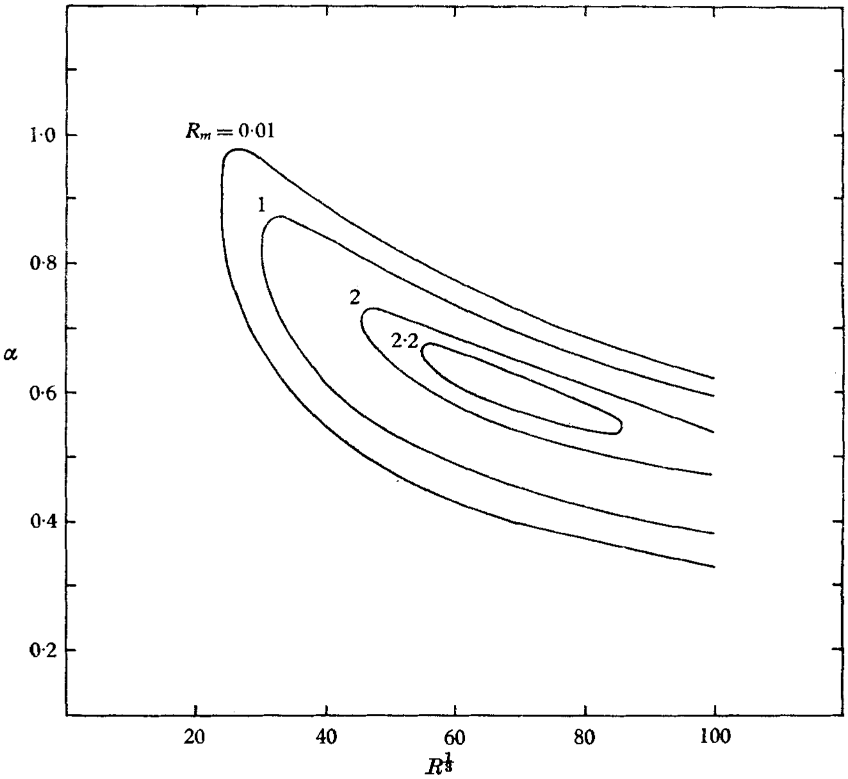

FIGURE 1. Neutral stability curve for $A^2 = 0{\cdot}05$ based on two-dimensional disturbances.

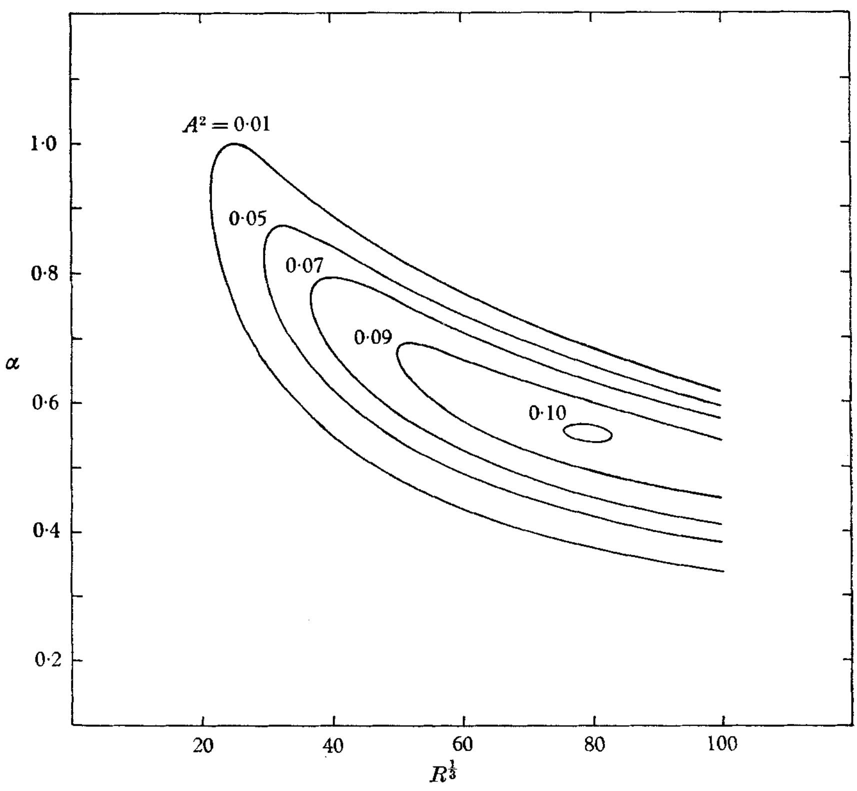

FIGURE 2. Neutral stability curve for $R_m = 1$ based on two-dimensional disturbances.

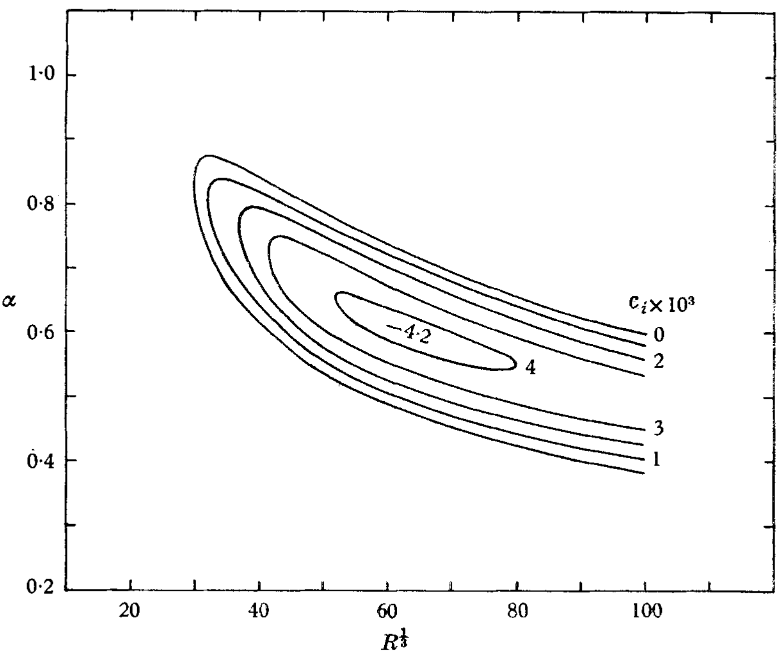

FIGURE 3*a*. Stability curve with amplification rates for $R_m = 1$ and $A^2 = 0{\cdot}05$.

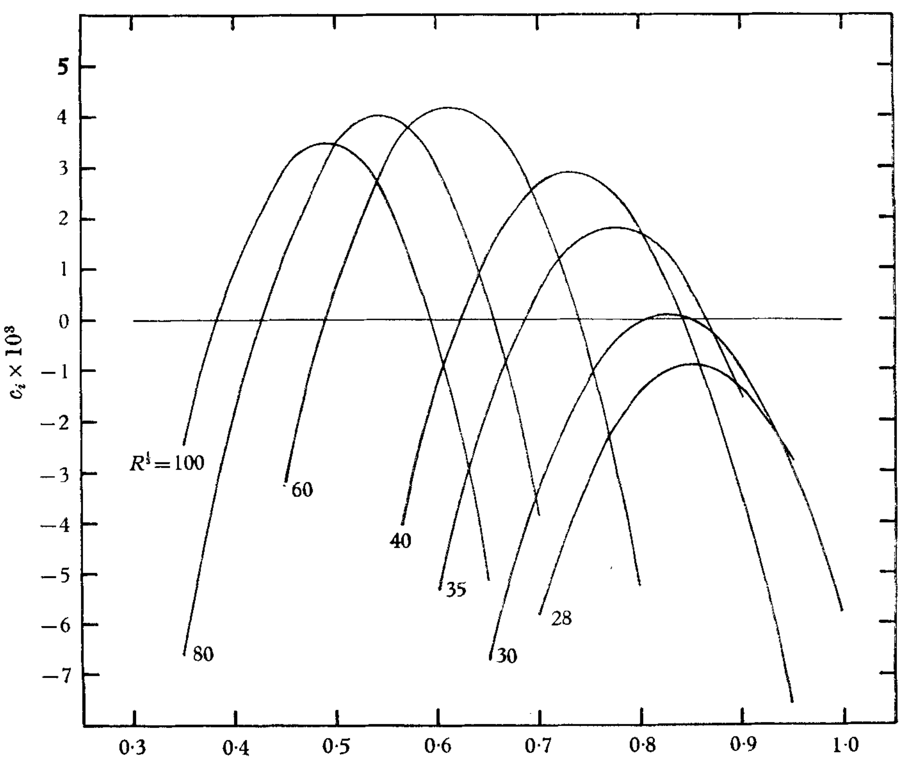

FIGURE 3*b*. Amplification rates for $R_m = 1$ and $A^2 = 0{\cdot}05$.

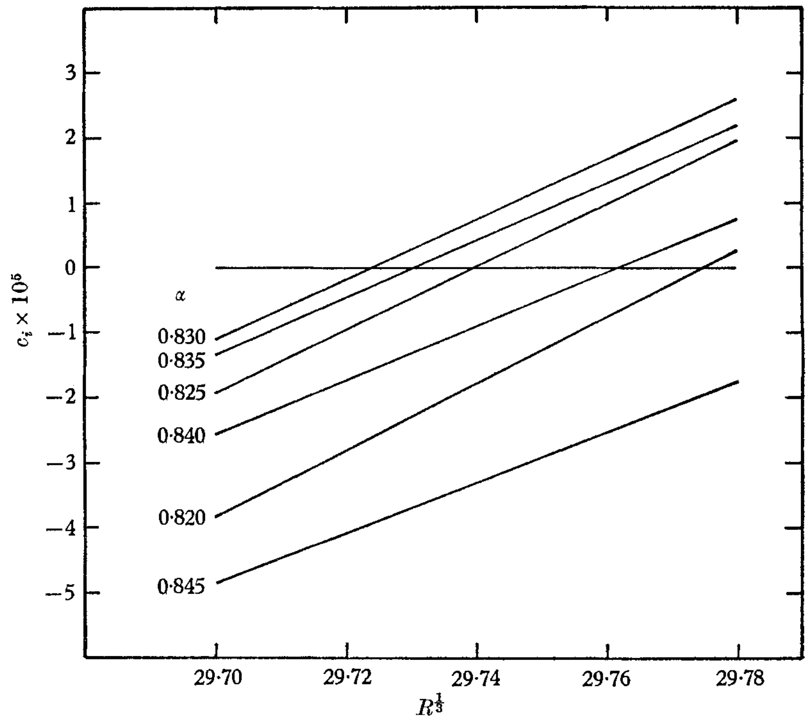

FIGURE 3*c*. Determination of the critical Reynolds number for $R_m = 1$ and $A^2 = 0{\cdot}05$.

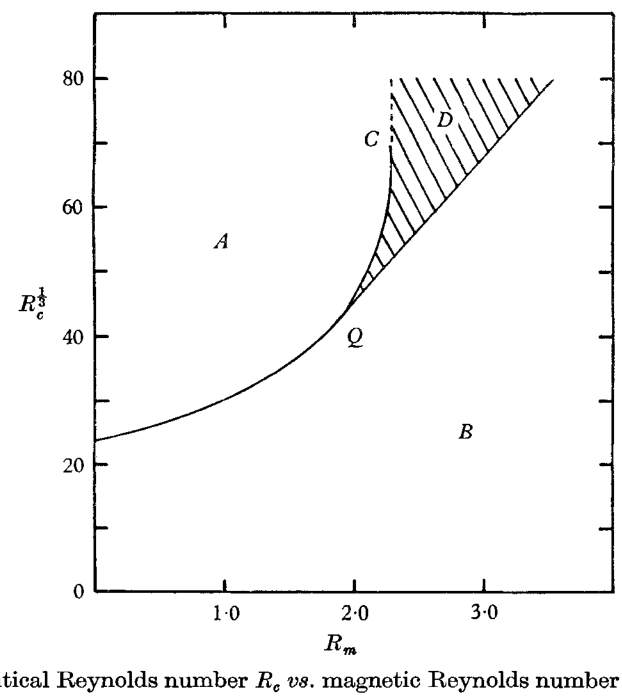

FIGURE 4. Critical Reynolds number R_c *vs*. magnetic Reynolds number R_m for $A^2 = 0{\cdot}05$. Region A, unstable for all disturbances. Region B, stable for all disturbances. Region D, stable for two-dimensional disturbances, unstable for three-dimensional disturbances (shaded portion).

based on two-dimensional disturbances. For two-dimensional disturbances the stability curve closes as R_m increases and finally degenerates to a point for $2{\cdot}2 < R_m < 2{\cdot}3$ with $A^2 = 0{\cdot}05$, which means that the flow can be completely stabilized by only considering two-dimensional disturbances. A tangent is drawn from the origin to the curve C, which touches where $R_m = 1{\cdot}9$. The critical Reynolds numbers follow the curve C for $R_m < 1{\cdot}9$ and follow the straight line

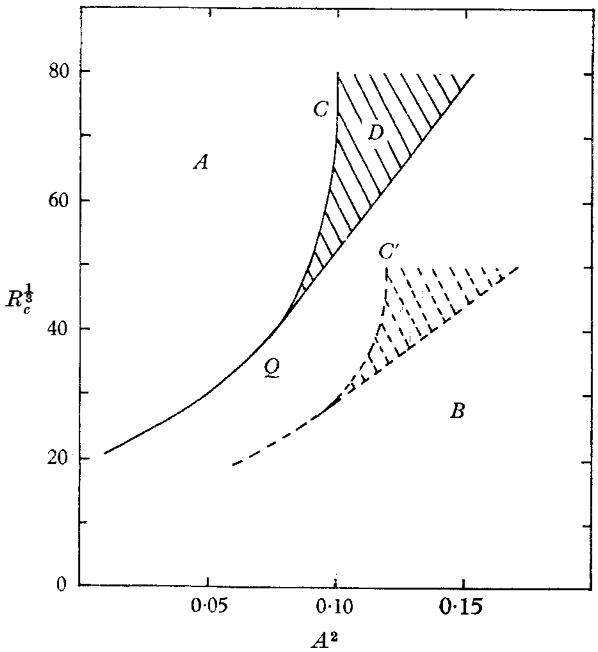

FIGURE 5. Critical Reynolds number R_c *vs.* square of Alfvén number A^2 for $R_m = 1$. Region A, unstable for all disturbances. Region B, stable for all disturbances. Region D, stable for two-dimensional disturbances, unstable for three-dimensional disturbances (shaded portion). C, present result. C', Tarasov's result.

for $R_m > 1{\cdot}9$, respectively (see Hunt (1966) for details). Therefore, results obtained for two-dimensional disturbances are significant only for $R_m < 1{\cdot}9$ with $A^2 = 0{\cdot}05$. Then, the effect of three-dimensional disturbances will become apparent at the value of $R_m = 1{\cdot}9$ for $A^2 = 0{\cdot}05$. Following the same criterion, we can also deduce that the effect of three-dimensional disturbances will become apparent at the value of $A^2 = 0{\cdot}075$ for $R_m = 1$.

5. Concluding remarks

A study has been made of the stability of a viscous, incompressible fluid with a finite conductivity flowing between parallel planes under a parallel magnetic field. The general form of the stability equation for a finite magnetic Reynolds number R_m is a sixth-order differential equation in terms of the magnetic field perturbation ψ alone. For small R_m the stability equation can be reduced to a fourth-order differential equation in terms of the velocity perturbation v alone.

For small R_m the neutral stability curves were obtained by Stuart (1954) and Hains (1965). The neutral stability curves for $R_m \lesssim 1$ and $R_m \cong \infty$ were obtained by Tarasov (1960) and Velikhov (1959), respectively. Both Velikhov (1959) and Tarasov (1960) employed an asymptotic method, in particular, Lin's (1955) method by Velikhov and Tollmien's (1936) method by Tarasov for the solution of the sixth-order differential equation. In the present investigation the complete sixth-order differential equation was numerically solved as an eigenvalue problem without the usual assumptions pertaining to the magnitudes of the hydrodynamic Reynolds number and the magnetic Reynolds number. The result obtained by Tarasov is compared with the present result for $R_m = 1$ (see figure 5). Tarasov's result presented in figure 5 has been obtained from the stability curve shown in his paper (1960). As can be seen in figure 5, the minimum critical Reynolds number R_c rises as A^2 increases. For $R_m = 1$, the neutral stability curve becomes a point for $0{\cdot}10 < A^2 < 0{\cdot}101$ in the present investigation, and for $0{\cdot}12 < A^2 < 0{\cdot}13$ in Tarasov's study. Although the minimum critical Reynolds number shows some discrepancies, Tarasov's result agrees qualitatively with the present one as far as the stabilizing influence is concerned. It is noticed that results obtained for small R_m by Stuart (1954) and Hains (1965) also agree qualitatively with the present result regarding the stabilizing influence. However, it should be noted that all the solutions made in these investigations are based on the ground that Squire's theorem can be extended to this case.

In the present case results obtained for two-dimensional disturbances have been modified to take into account three-dimensional disturbances using the criterion demonstrated by Hunt (1966). We have obtained the region where two-dimensional disturbances are completely stable, but three-dimensional disturbances are found to be the most unstable by this criterion (see figures 4 and 5). It might be very interesting to find the most unstable disturbances with some angle θ with respect to the flow direction. However, it is quite laborious to investigate all the possible three-dimensional disturbances giving the most unstable mode. In practice, this region can be excluded by limiting the critical Reynolds number to the point Q ($R_m = 1{\cdot}9$ for $A^2 = 0{\cdot}05$ in figure 4 and $A^2 = 0{\cdot}075$ for $R_m = 1$ in figure 5). For any combination of R_m and A^2, we can always find a point Q. This immediately eliminates the necessity for the investigation of three-dimensional disturbances. In an experiment with a conducting fluid, an applied magnetic field can be constrained to give the proper combination of R_m and A^2. Then, the effect of three-dimensional disturbances will not be apparent according to the criterion.

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REFERENCES

DRAZIN, P. G. 1960 *J. Fluid Mech.* **8**, 130.

HAINS, F. D. 1965 *Phys. Fluids*, **8**, 2014.

HUNT, J. C. R. 1966 *Proc. Roy. Soc.* A **293**, 342.

LIN, C. C. 1955 *Theory of Hydrodynamic Stability*. Cambridge University Press.

MICHAEL, D. H. 1953 *Proc. Camb. Phil. Soc.* **49**, 166.

SQUIRE, H. B. 1933 *Proc. Roy. Soc.* A **142**, 621.

STUART, J. T. 1954 *Proc. Roy. Soc.* A **221**, 189.

TARASOV, YU. A. 1960 *Soviet Phys. JETP* **10**, 1209 (in English).

THOMAS, L. H. 1953 *Phys. Review*, **91**, 780.

TOLLMIEN, W. 1936 *NACA TM*, 792.

VELIKHOV, E. P. 1959 *Soviet Phys. JETP* **9**, 848 (in English).

WOOLER, P. T. 1961 *Phys. Fluids*, **4**, 24.